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TRAVELING WAVES IN THE SUPERSONIC FLUTTER PROBLEM OF PANELS OF FINITE LENGTH

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Some peculiarities are considered of the shapes of natural plate oscillations, which may be defined by means of the well-known equation

$$D\left(\frac{\partial^4 W}{\partial x^4} + 2\frac{\partial^4 W}{\partial x^2 \partial y^2} + \frac{\partial^4 W}{\partial y^4}\right) + \mu\frac{\partial^2 W}{\partial t^2} = N_1\frac{\partial^2 W}{\partial x^2} + N_2\frac{\partial^2 W}{\partial y^2} + Q \qquad (1)$$

in the assumption that a plate of thickness h occupies a region shaped as an unlimited strip $0 \le x \le a$ or a rectangle $0 \le x \le a$, $0 \le y \le b$, and the load Q is given by the well-known formula of the piston theory[1]

$$Q = \frac{p_0 \varkappa}{c_0}\left(c\frac{\partial W}{\partial x} - \frac{\partial W}{\partial t}\right) \qquad (2)$$

(the plate moves in a gas in the direction of the positive axis x). Then the following dimensionless quantities are introduced

$$\frac{x}{a}, \frac{y}{b}, \frac{W}{h}, t/\sqrt{\frac{\mu a^4}{D}} \qquad (3)$$

for which the former designations x, y, w, t, are retained, and the perturbations of the plate are considered, represented as

$$W(x, y, t) = w(x, t)f(y) \qquad (4)$$

$$f(y) \equiv 1 \quad \text{or} \quad f(y) = \sin \pi n y, \quad n = 1, 2, \cdots$$
$$\qquad (5)$$
$$w(x, t) = X(x)e^{wt}$$

The right-hand side of Eq. (5), generally speaking, may prove to be the complex function of the real variables x, t, for example, as a result of the complexity of the frequency ω

$$\omega = p + iq \qquad (6)$$

In Eq. (4) instead of $W(x, t)$ we substitute the functions

$$w(x, t) = ReX(x)e^{wt}, \qquad w(x, t) = ImX(x)e^{wt} \tag{7}$$

and obtain the real perturbations, which at $q \neq 0$ we designate as the natural plate oscillations. With the selection of the functions $f(y)$ assumed, the peculiarities of the natural oscillations of interest to us are determined by the functions [Eq. (7)], which, therefore, we shall also designate as the natural plate oscillations.

It is easy to see that every function $X(x)$, included in the right-hand side of [Eq. (5)], is an eigenfunction (nontrivial solution) of the boundary-value problem

$$\frac{d^4X}{dx^4} - 2k\pi^2\frac{d^2X}{dx^2} + k^2\pi^4X - A\frac{dX}{dx} = \lambda^0 X \tag{8}$$

$$L_i(X) = 0, \qquad i = 1, 2, 3, 4$$

where $L_i(X) = 0$ designate the linear homogeneous boundary conditions on the edges $x = 0$, $x = 1$, and the dimensionless parameters k, A, λ^0, are determined by the equalities

$$k = \frac{n^2a^2}{b^2} + \frac{1}{2}n_1, \qquad n_1 = \frac{a^2N_1}{\pi^2D}, \qquad n_2 = \frac{a^2N_2}{\pi^2D}$$

$$A = \frac{a^3\rho_0\varkappa}{\mathcal{D}c_0}c, \qquad \lambda^0 = \lambda + \pi^4\left[\frac{1}{4}n_1^2 + \frac{n^2a^2}{b^2}(n_1 - n_2)\right] \tag{9}$$

$$\lambda = -(\omega^2 + B\omega), \qquad B = \frac{a^2}{\sqrt{\mathcal{D}\mu}}\frac{\rho_0\varkappa}{c_0}$$

Combining

$$X(x) = \sum_{j=1}^{4} C_j X^j(x, k, A, \lambda^0) \tag{10}$$

of the four linearly independent solutions $X^j(x, k, A, \lambda^0)$ of Eq. (8) and satisfying the boundary conditions

$$\sum_{j=1}^{4} L_i(X^j)C_j = 0, \qquad i = 1, 2, 3, 4 \tag{11}$$

we obtain for the determination of the eigenvalues λ^0 the equation

$$F(k, A, \lambda^0) \equiv \begin{vmatrix} L_1(X^1) & L_1(X^2) & L_1(X^3) & L_1(X^4) \\ L_2(X^1) & L_2(X^2) & L_2(X^3) & L_2(X^4) \\ L_3(X^1) & L_3(X^2) & L_3(X^3) & L_3(X^4) \\ L_4(X^1) & L_4(X^2) & L_4(X^3) & L_4(X^4) \end{vmatrix} = 0 \tag{12}$$

the left-hand side of which is an integral analytical function of the parameters k, A, λ^0. To each root λ^0 of Eq. (12) corresponds at least one solution C_j of the

system [Eq. (11)] and the eigenfunction [Eq. (10)]. The frequencies ω correspond-
ing to this function are found from the quadratic equation $\omega^2 + B\omega + \lambda = 0$

$$\omega = -\frac{B}{2} \pm \sqrt{\frac{B^2}{4} - \lambda} \tag{13}$$

If the eigenvalue λ^0 is a simple root of Eq. (12), one eigenfunction $X(x)$
corresponds to it (to within the constant). If the eigenvalue λ^0 is a double root
of Eq. (12) and the matrix rank of Eq. (12) is equal to two, two linearly inde-
pendent functions correspond to the eigenvalue λ^0. If however, the matrix
rank of Eq. (12) is equal to three, only one eigenfunction $X(x)$ corresponds to
the eigenvalue λ^0. In the latter case, for the sake of completeness, together
with the solution [Eq. (5)], it is also necessary to consider the solution

$$w(x, t) = [X_{(1)}(x) + tX(x)]e^{wt} \tag{14}$$

where $X_{(1)}(x)$ is a function, adjoint to the eigenfunction $X(x)$.[2,3] For example,
for a hinged panel with $k = -2.5$, $A = 0$ to the double eigenvalue

$$\lambda_1{}^0 = \lambda_2{}^0 = \frac{9}{4}\pi^4 \tag{15}$$

correspond two eigenfunctions[4]

$$X_1(x) = \sin \pi x, \qquad X_2(x) = \sin 2\pi x \tag{16}$$

An example of employing adjoint functions is given in Ref. 5.
    Every eigenfunction $X(x)$ of the boundary problem [Eq. (8)] may be presented
as

$$X(x) = X_1(x) + iX_2(x) \tag{17}$$

where $X_1(x)$, $X_2(x)$ are real functions (the case of the equality to zero of one
of the functions $X_1(x)$ or $X_2(x)$ is not excluded.) The zeros of the eigenfunction
[Eq. (17)] separate the segment $0 \leq x \leq 1$ into a finite number of intervals, in
each of which the function [Eq. (17)] is represented trigonometrically as

$$X(x) = |X(x)|e^{i\psi(x)} = |X(x)|[\cos \psi(x) + i \sin \psi(x)]$$
$$|X(x)| = \sqrt{X_1{}^2(x) + X_2{}^2(x)} \tag{18}$$

$$\cos \psi(x) = X_1(x)/|X(x)|, \qquad \sin \psi(x) = X_2(x)/|X(x)| \tag{19}$$

In each of these intervals the argument $\psi(x)$ is a continuously differentiable
function x; $\psi(x)$ is not defined in the interval boundary points and may have
discontinuities.
    Assume the functions $X_1(x)$, $X_2(x)$ are linearly dependent in the eigenfunction
[Eq. (17)]. In this case we designate the eigenfunction as a real function (con-
ditionally). It is easy to see that the relation $\psi(x) = \text{Const.}$ is satisfied for the
real (as indicated) eigenfunction at every interval of the continuity $\psi(x)$.

Assume the functions $X_1(x)$, $X_2(x)$ are linearly independent in the eigenfunction [Eq. (17)]. In this case we designate the eigenfunction as a complex function. For the complex (as indicated) eigenfunction, the relation $\psi(x) \neq$ Const. is satisfied at every interval of the continuity $\psi(x)$. Actually, otherwise at some of the intervals of the continuity $\psi(x)$ the following relation would be satisfied

$$X(x) = X_1(x) + {}_iX_2(x) \equiv |X(x)| \ (c_1 + ic_2)$$

where $c_1 = \cos \psi(x)$ and $c_2 = \sin \psi(x)$ are real constants, not simultaneously equal to zero, and the functions

$$X_1(x) = c_1|X(x)|, \qquad X_2(x) = c_2|X(x)|$$

would prove to be linearly dependent in the examined interval, which comprises a part of the span $0 \leq x \leq 1$. This, however, is impossible, since the functions $X_1(x)$, $X_2(x)$ are assumed to be linearly independent in the span $0 \leq x \leq 1$ and are solutions of the same linearly ordinary differential [Eq. (6)].

Substituting Eqs. (6) and (8) in Eq. (5), we obtain

$$w(x, t) = |X(x)|e^{i\psi(x)+(p+iq)t}$$

It is obvious from here that the natural oscillations [Eq. (7)] are described by the equalities

$$w(x, t) = |X(x)| \cos [\psi(x) + qt]e^{pt}$$

$$w(x, t) = |X(x)| \sin [\psi(x) + qt]e^{pt}$$

(20)

The relation $\psi(x) =$ Const. is satisfied for the real eigenfunction $X(x)$ at the intervals of the continuity $\psi(x)$, which gives grounds in this case to refer to the natural oscillations [Eq. (20)] as standing waves.

For the complex eigenfunction $X(x)$ the relation $\psi(x) \neq$ Const. is satisfied in the intervals of the continuity $\psi(x)$, as proven. The analogy with the functions

$$\cos (Cx + qt) \qquad \sin (Cx + qt) \qquad (C = \text{Const.}) \qquad (21)$$

in this case gives grounds to refer to the natural oscillations [Eq. (20)] as traveling waves. The propagation velocity of the traveling waves [Eq. (21)] is given by the formula

$$V = -q/C \qquad (22)$$

The dimensionless propagation velocity of the traveling waves, Eq. (20), is determined from the formula

$$V(x) = -q/(d\psi(x)/dx) \qquad (23)$$

The dimensional velocity is obtained by multiplying Eq. (23) by

$$a/\sqrt{\frac{\mu a^4}{\mathfrak{D}}}$$

Unlike Eq. (22), the velocity of the traveling waves [Eq. (23)] at $d\psi(x)/dx \neq$ Const. is different at different points of the plate.

Only the real eigenfunction $X(x)$ and the natural oscillations, Eq. (20), in the form of standing waves can correspond to the real single eigenvalue λ^0 of the boundary problem [Eq. (8)]. To the real eigenvalue λ^0 may correspond the complex eigenfunction [Eq. (17)] when, and only when, λ^0 is a multiple root of Eq. (12) and when at least two linearly independent eigenfunctions correspond to it. Equations (15), (16) may serve as illustrations; the linear combination

$$X(x) = 0.5 \sin 2\pi x + i \sin \pi x$$

is the complex eigenfunction, corresponding to the real eigenvalue [Eq. (15)], for which from Equations (19), (20) we obtain (with $q \neq 0$) the natural oscillations in the form of waves traveling on the plate.

$$w(x, t) = \sin \pi x \sqrt{1 + \cos^2 \pi x} \cos [\psi(x) + qt]e^{pt}$$

$$w(x, t) = \sin \pi x \sqrt{1 + \cos^2 \pi x} \sin [\psi(x) + qt]e^{pt}$$

$$\cos \psi(x) = \frac{\cos \pi x}{\sqrt{1 + \cos^2 \pi x}}, \qquad \sin \psi(x) = \frac{1}{\sqrt{1 + \cos^2 \pi x}}$$

The graph of the function $\psi(x)$ is shown in Fig. 1.

The conditionless complex eigenfunction [Eq. (17)] and the natural oscillations [Eq. (20)] in the form of waves traveling on a plate correspond to the complex eigenvalue λ^0 of the boundary problem [Eq. (8)] ($\lambda^0 = \lambda_{(1)}^0 + i\lambda_{(2)}^0$, $\lambda_{(2)}^0 \neq 0$).

The flutter of the plates (natural oscillations [Eq. (20)] in the case $p > 0$), detectable on the basis of Eqs. (1) and (2), is possible only for the complex eigenvalues λ^0 of the boundary problem [Eq. (8)], which actually exist with corresponding values of velocity A[4]. Consequently, natural oscillations in the form of traveling waves, encountered for real, as exceptions, are typical phenomena
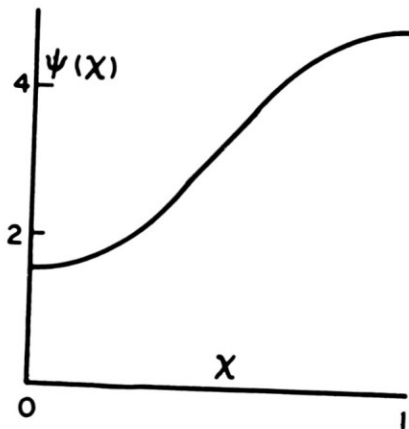

Fig. 1.

for plate flutter, which is confirmed by experiments.[7,8] The theoretical quantitative investigation of the character of traveling waves, Eq. (20), with flutter which are, as a rule, associated with extremely cumbersome computations, became possible relatively recently due to the employment of speed computers.

Except for the case when the matrix rank of Eq. (12) is less than three, the eigenfunction [Eq. (10)] may be found by means of the formula

$$X(x) = c_1 e^{-\alpha x} \sin \beta x + c_2 e^{-\alpha x} \cos \beta x$$
$$+ c_3 e^{\alpha x} \, sh \, \sqrt{\beta^2 - 2\alpha^2 + 2k\pi^2} \, x + c_4 e^{\alpha x} \, ch \, \sqrt{\beta^2 - 2\alpha^2 + 2k\pi^2} \, x \quad (24)$$

For the case of hinged edges $x = 0, 1$ the constants α, β satisfy the system of Equations (22), (23) in Ref. 4, and the constants C_j have the value

$$c_1 = e^{\alpha} \, [\alpha \, \sqrt{\beta^2 - 2\alpha^2 + 2k\pi^2} \, ch \, \sqrt{\beta^2 - 2\alpha^2 + 2k\pi^2}$$
$$+ (\alpha^2 - \beta^2 - k\pi^2) \, sh \, \sqrt{\beta^2 - 2\alpha^2 + 2k\pi^2}]$$
$$- \alpha \, \sqrt{\beta^2 - 2\alpha^2 + 2k\pi^2} \, e^{-\alpha} \cos \beta \quad (25)$$

$$c_2 = \alpha(e^{-\alpha} \, \sqrt{\beta^2 - 2\alpha^2 + 2k\pi^2} \sin \beta - sh \, \sqrt{\beta^2 - 2\alpha^2 + 2k\pi^2} \, e^{\alpha} \cdot \beta)$$

$$c_3 = e^{-\alpha}[(\beta^2 - \alpha^2 + k\pi^2) \sin \beta - \alpha\beta \cos \beta] + \alpha\beta e^{\alpha} \, ch \, \sqrt{\beta^2 - 2\alpha^2 + 2k\pi^2}$$

$$c_4 = - c_2$$

For the case of clamped edges $x = 0,1$ Eq. (2.2) in Ref. 4 should be replaced by the equation given in Ref. 4 on p. 242, and the constants C_j have the value

$$c_1 = (e^{2\alpha} \, ch \, \sqrt{\beta^2 - 2\alpha^2 + 2k\pi^2} - \cos \beta) \, \sqrt{\beta^2 - 2\alpha^2 + 2k\pi^2}$$
$$- 2\alpha e^{2\alpha} \, sh \, \sqrt{\beta^2 - 2\alpha^2 + 2k\pi^2}$$

$$c_2 = \sqrt{\beta^2 - 2\alpha^2 + 2k\pi^2} \sin \beta - \beta e^{2\alpha} \, sh \, \sqrt{\beta^2 - 2\alpha^2 + 2k\pi^2} \quad (26)$$

$$c_3 = \beta \cos \beta + 2\alpha \sin \beta - \beta e^{2\alpha} \, ch \, \sqrt{\beta^2 - 2\alpha^2 + 2k\pi^2}$$

$$c_4 = - c_2$$

The process of finding the eigenfunctions consists, first of all, in the solution of quite complex transcendental equations with respect to the parameters α, β (which are complex numbers for a complex λ^0) and in subsequent computations according to formulas of the type (24), (25), and (26). In the formulas indicated it is advantageous to regroup terms in order to preserve accuracy, which is lost during the subtraction of numbers large in modulus but differing little between themselves.

Below are presented results of the computation of oscillations of a plate clamped at the edges $x = 0, 1$, in the case $k = 0$.

The real eigenfunctions, corresponding with $A = 0$ to the first two single eigenvalues $\lambda_1^0 = 500.6$, $\lambda_2^0 = 3804$, are depicted in Fig. 2; the corresponding natural oscillations appear as standing waves. With $A = 636.6$, only one eigenfunction $X(x)$, depicted in Fig. 2, corresponds to the coinciding eigenvalues $\lambda_1^0 = \lambda_2^0 = 2741$; the values of this function in twelve points are presented in column 1 of Table I; the corresponding natural oscillations appear as standing waves. The double eigenvalue $\lambda_1^0 = \lambda_2^0 = 2741$ has also the adjoint function $X_{(1)}(x)$, corresponding to it, as well as the solution (14); the form of this function is not determined by us.

With $A > 636.6$ the first two eigenvalues λ_1^0, λ_2^0 become complex conjugated numbers. Figure 3 depicts the real (on the left) and imaginary (on the right) parts of the complex eigenfunction $X(x)$, corresponding to the first eigenvalue $\lambda_1^0 = 2866 - i\,772.7$ at $A = 700$ and to the first eigenvalue $\lambda_1^0 = 137,511 - i\,206,656$ at $A = 20,000$. The values $ReX(x)$, $ImX(x)$ for $A = 700$ are given in columns 2 and 3, and for $A = 20,000$ in columns 4 and 5 in Table I.

Figures 4 and 5 depict the functions $|X(x)|$, $\psi(x)$, computed according to Eq. (19) (each function $|X(x)|$ was additionally normalized). The corresponding values of the frequencies $\omega_1 = p_2 + iq_1$, $\omega_2 = p_2 + iq_2$, computed according to the formula (13) for the case $\lambda = \lambda^0$ [see designations (9)], are given in Table II ($q_2 = -q_1$).

The data in Fig. 5 and Table II show that the flutter ($p_1 > 0$) is realized as waves (20), traveling in the direction of the negative axis x (downward with the flow, $q_1 > 0$); the amplitude of the waves traveling toward the flow ($q_2 < 0$) attenuates ($p_2 < 0$). The shape $w(x, t)$, which is assumed by the wave traveling downward with the flow ($q_1 > 0$) [Eq. (20)] at the instants

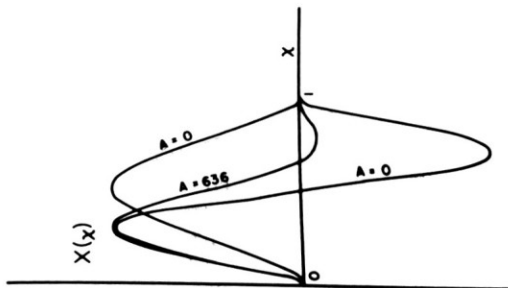$$t = \frac{2\pi}{q} \cdot \frac{m}{40}, \qquad m = 0, \cdots, 39$$

Fig. 2.

x	№ 1	№ 2	№ 3	№ 4	№ 5
0.0	0	0	0	0	0
0.01	0.0048	0.0047	-0.0036	0.0305	-0.0354
0.1	0.3393	0.3347	-0.2498	1	-0.6780
0.2	0.8303	0.8201	-0.5832	0.5904	0.5065
0.3	1	1	-0.6442	-0.2270	0.4213
0.4	0.7983	0.8293	-0.4324	-0.2181	-0.0935
0.5	0.4216	0.4892	-0.1183	0.0268	-0.0978
0.6	0.0920	0.1792	0.1046	0.0403	0.0054
0.7	-0.0753	0.0027	0.1715	-0.0003	0.0143
0.8	-0.0904	-0.0433	0.1210	-0.0041	0.0049
0.9	-0.0352	-0.0208	0.0388	-0.0002	-0.0008
1.0	0	0	0	0	0

Таблица 2

B	0	10	20	30	40	50
$p_1(700)$	7.2	2.2	-2.7	-7.6	-12.3	-17.0
$q_1(700)$	54.0	53.8	53.1	51.9	50.2	48.0
$p_2(700)$	-7.2	-12.2	-17.3	-22.4	-27.7	-33.0
$p_1(20000)$	235.4	230.4	225.4	220.4	215.4	210.6
$q_1(20000)$	439.2	439.2	439.2	439.0	438.7	438.6
$p_2(20000)$	-235.4	-240.4	-245.4	-250.4	-255.4	-260.6

Fig. 3a.
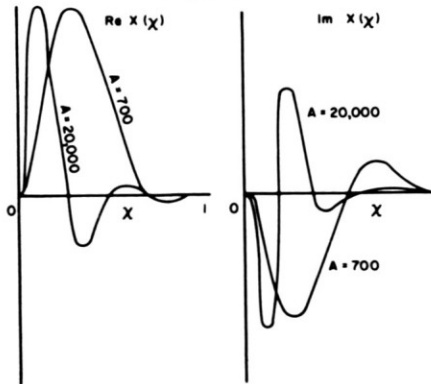
Fig. 3b.
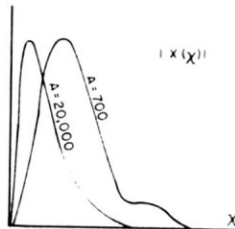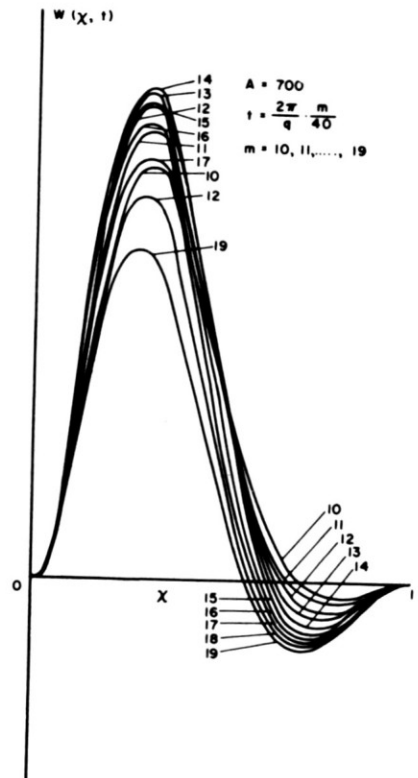

Fig. 4.
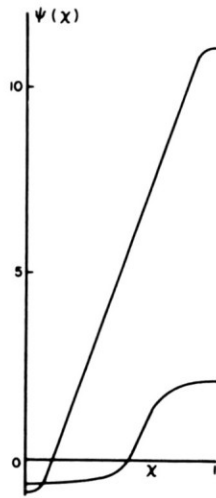
Fig. 5.
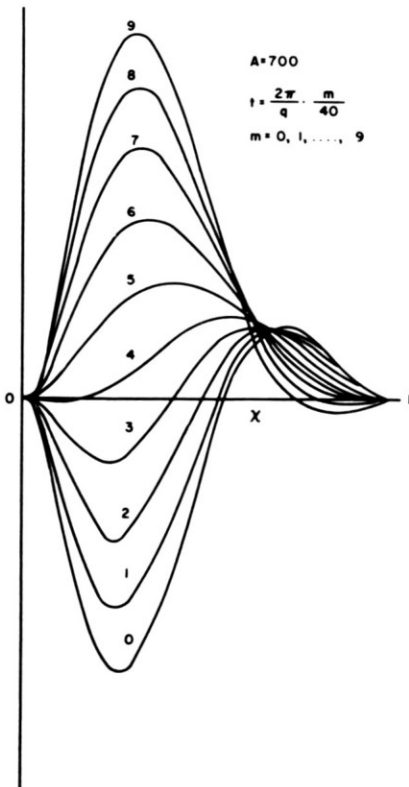
Fig. 6.

Fig. 7.

is presented for $A = 700$ in Fig. 6 ($m = 0$-9), Fig. 7 ($m = 10$-19), Fig. 8 ($m = 20$-29), Fig 9 ($m = 30$-39); for the convenience of comparing the different shapes (20) the multiplier e^{pt} was dropped. Analogous results for $A = 20{,}000$ are represented in Figs. 10-13. A visual representation of the character of traveling waves is given by a multiplication film, obtained by means of sequential exposure of curves shown in Figs. 6-13.

The character of the theoretically obtained traveling waves, whose maximum amplitude shifts sharply toward the rear boundary of the plate with rising velocity A, explains the results of the experiments,[8] in which the destruction of the panels due to flutter always started at the rear boundary.

Attention is drawn to the large difference in the distribution of amplitudes of the traveling waves (20), satisfying clamped edge conditions, and monochromatic waves, Eq. (21), which do not satisfy the clamped boundary conditions,
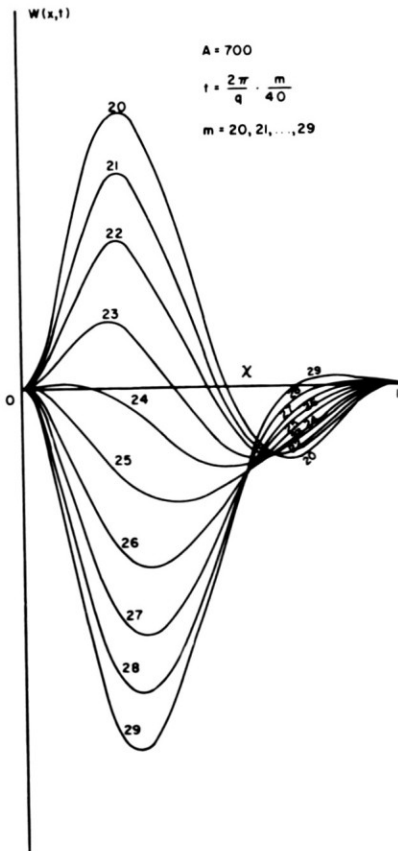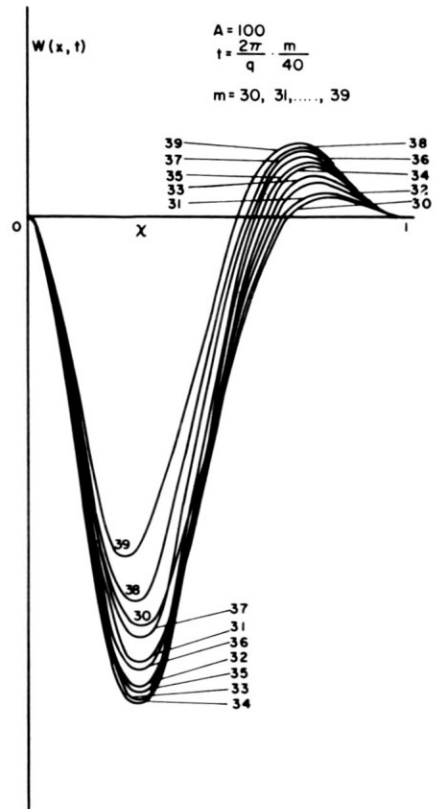

Fig. 8.

Fig. 9.

but which, nevertheless, are employed in some investigations of the flutter of plates and shells. It seems probable that this circumstance is not related to the character of the aerodynamic theory applied.
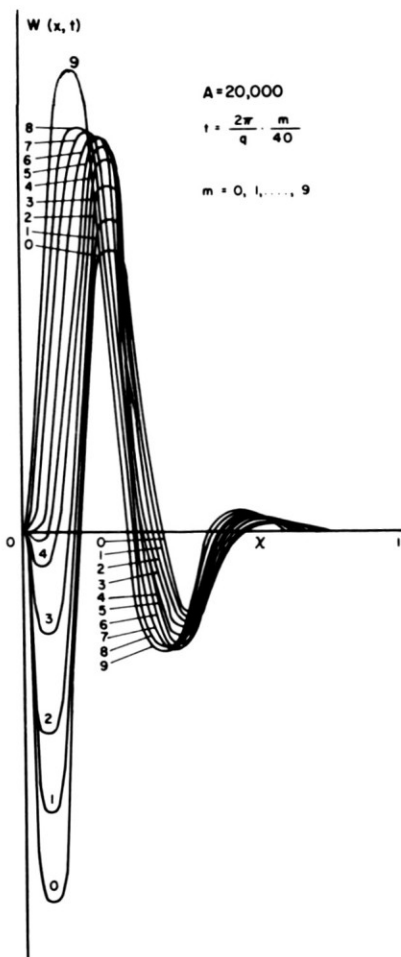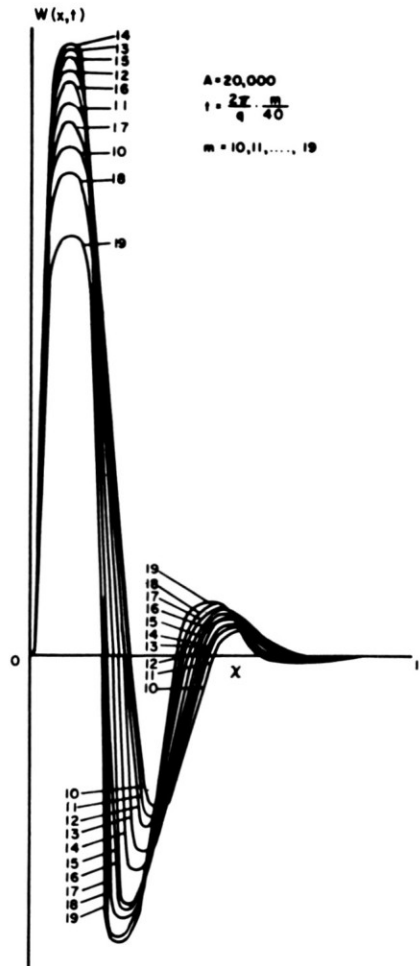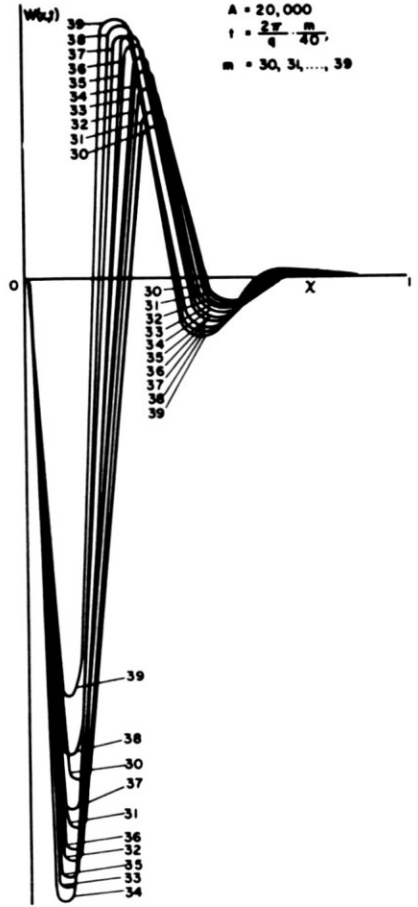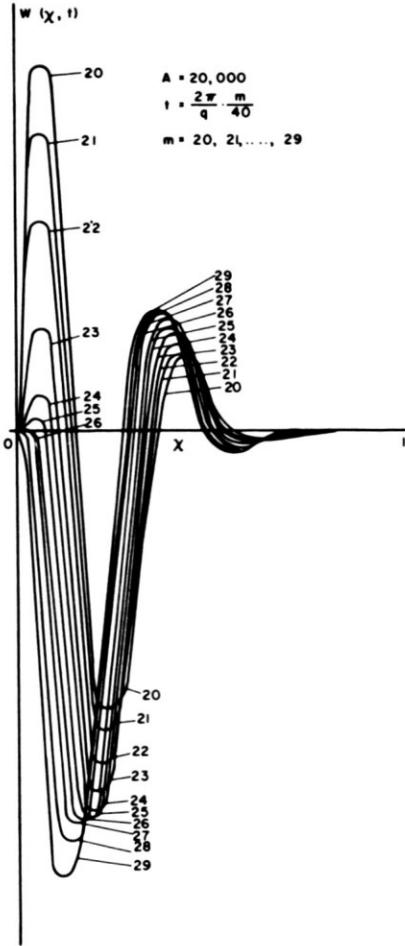
Fig. 10.

Fig. 11.

Fig. 12.

Fig. 13.

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